Solutions

Exam 1
Chapter 1 and 2.1

Name:____________________

Do not write your name on any other page. Answer the following questions. *Answers without proper evidence of knowledge will not be given credit.* Make sure to make reasonable simplifications. Do not approximate answers. Give exact answers. **Only scientific calculators are allowed on this exam.**

Show your work!

1. (5 points) Verify that $y = x\cos x$ is a solution to the differential equation

$$y' + y\tan x = \cos x.$$

$$y' = \cos x - x\sin x$$

Thus

$$y' + y\tan x = \cos x - x\sin x + x\cos x\tan x$$

$$= \cos x - x\sin x + x\sin x$$

$$= \cos x \quad \checkmark$$

2. (5 points) A diesel car gradually speeds up so that for the first 10 s its acceleration is given by

$$\frac{dv}{dt} = (0.12)t^2 + (0.6)t \quad (\text{ft/s}^2).$$

If the car starts from rest ($x_0 = 0$, $v_0 = 0$), find the distance it has traveled at the end of the first 10 seconds and its velocity at that time.

$$v(t) = \int (0.12)t^2 + (0.6)t \, dt = (0.04)t^3 + (0.3)t^2 + C$$

$$C = v_0 = 0$$

$$x(t) = \int (0.04)t^3 + (0.3)t^2 \, dt = (0.01)t^4 + (0.1)t^3 + C$$

$$C = x_0 = 0.$$

So

$$x(10) = (0.01)(10)^4 + (0.1)(10)^3 = 200 \text{ ft}$$

and

$$v(10) = (0.04)(10)^3 + (0.3)(10)^2 = 70 \text{ ft/s}$$

3. (10 points) Find the general solution to the differential equation

$$2\sqrt{x}\frac{dy}{dx} = \cos^2 y.$$

$$\int \frac{dy}{\cos^2 y} = \int \frac{dx}{2\sqrt{x}}$$

$$\int \sec^2 y\,dy = x^{1/2} + C$$

$$\tan y = x^{1/2} + C$$

$$y = \tan^{-1}(x^{1/2} + C).$$

4. A tank initially contains 60 gallons of pure water. Brine containing 1 lb or salt per gallon enters the tank at a rate of 2 gal/min, and the (perfectly mixed) solution leaves the tank at 3 gal/min.

(a) (2 points) When is the tank empty?

(b) (6 points) Find the amount of salt in the tank after t minutes.

(c) (2 points) What is the maximum amount of salt ever in the tank?

(a) $V(t) = 60 - t$ gal. Empty in 60 minutes.

(b) $\frac{dx}{dt} + \frac{3}{60-t} x = 2 \cdot 1$, $\rho(t) = e^{\int \frac{3}{60-t} dt} = e^{-3\ln(60-t)} = (60-t)^{-3}$

$$(60-t)^{-3} \cdot x = \int 2 \cdot (60-t)^{-3} dt = (60-t)^{-2} + C$$

$$x = (60-t) + C(60-t)^3$$

$$x(0) = 0 = 60 + C \cdot 60^3 \Rightarrow C = -60^{-2} = \frac{-1}{3600}.$$

and $x = (60-t) - \frac{(60-t)^3}{3600}$.

(c) $x' = -1 + \frac{3}{3600}(60-t)^2 \Rightarrow t = 60 \pm \sqrt{1200}$

Only $60 - \sqrt{1200}$ makes sense. It is a max.

$$x(60 - \sqrt{1200}) = \sqrt{1200} - \frac{(1200)^{3/2}}{3600} \approx 23.09 \text{ lbs of salt.}$$

5. (10 points) Find the general solution to the differential equation

$$(1 + ye^{xy})dx + (2y + xe^{xy})dy = 0.$$

$$\int 1+ye^{xy}\,dx = x + e^{xy} + C(y)$$

$$\int 2y+xe^{xy}\,dy = y^2 + e^{xy} + C(x)$$

$$F(x,y) = e^{xy} + x + y^2 = C.$$

6. (10 points) Consider a rabbit population satisfying the logistic equation

$$\frac{dP}{dt} = 2P - (0.005)P^2.$$

If the initial population is 120 rabbits, how many months does it take for $P(t)$ to reach 95% of its limiting population M?

$$\frac{dP}{dt} = 2P - (0.005)P^2 = (0.005)P(400-P)$$

Logistic Eqn Solution

$$P(t) = \frac{120 \cdot 400}{120 + 280 \cdot e^{-2t}}.$$

$$380 = \frac{48000}{120 + 280e^{-2t}}$$

$$126.32 = 120 + 280e^{-2t}$$

$$6.32 = +280e^{-2t}$$

$$+0.0226 = e^{-2t}$$

$$t \approx \frac{\ln(0.0226)}{-2} \approx 1.896 \text{ years.}$$